ON A FRACTIONAL SUBLINEAR ELLIPTIC EQUATION WITH A VARIABLE COEFFICIENT

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Abstract. We study existence and uniqueness of bounded solutions to a fractional sublinear elliptic equation with a variable coefficient, in the whole space. Existence is investigated in connection to a certain fractional linear equation, whereas the proof of uniqueness relies on uniqueness of solutions to an associated fractional porous medium equation with variable density.

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1 Introduction

We are concerned with existence and uniqueness of bounded solutions to the following fractional sublinear equation:

$$(-\Delta)^{\frac{\sigma}{2}} u = \rho\, u^{\alpha} \qquad \text{in } \mathbb{R}^N. \tag{1.1}$$

The nonlocal operator $(-\Delta)^{\frac{\sigma}{2}}$ is the fractional Laplacian of order $\sigma/2$ with $\sigma \in (0,2), N \geq 2$. Thus the following representation in terms of a singular integral holds:

$$(-\Delta)^{\sigma/2} g(x) = C_{N,\sigma}\ \text{P.V.} \int_{\mathbb{R}^N} \frac{g(x)-g(z)}{|x-z|^{N+\sigma}} \mathrm{d}z, \tag{1.2}$$

for any g belonging to the Schwartz class, where $C_{N,\sigma}$ is an appropriate positive normalization constant depending on N and σ (see (3.8)). The function ρ is nonnegative and bounded in $\mathbb{R}^N$, and decays at infinity fast

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enough. If we replace the nonlocal operator in (1.1) by the Laplace operator Δ, then we obtain the following sublinear elliptic equation:

$$-\Delta u = \rho\, u^{\alpha} \quad \text{in } \mathbb{R}^N\,, \tag{1.3}$$

which, together with its counterpart in bounded domains of $\mathbb{R}^N$ completed with Dirichlet boundary conditions, has been extensively studied in the literature (see [3], [4], [10], [11], [13], [14], [17]). In particular, in [3], existence and uniqueness of bounded solutions to equation (1.3) have been established, under the assumption $\rho \in L^{\infty}_{loc}(\mathbb{R}^N)$, $\rho \geq 0$. More precisely, in [3] it has been shown that existence of solutions to problem (1.3) holds if and only if the linear problem

$$-\Delta U = \rho \qquad \text{in } \mathbb{R}^N \tag{1.4}$$

admits a bounded solution; moreover, the solution is unique in the class of solutions v satisfying $\liminf_{|x|\to\infty} v(x) = 0\,.$ Whereas, asymptotic behavior as $|x| \to \infty$ of solutions to equation (1.3) has been addressed in [10], [11] and [14], under appropriate assumptions on ρ.

Recently, also the analysis of fractional semilinear elliptic equations have attracted the attention of various authors (see, *e.g.*, [1], [2], [5], [6]). In particular, for further references we point out that in [1], [2] existence and multiplicity of solutions have been studied for the problem

$$\begin{cases} (-\Delta)^{\frac{\sigma}{2}} u = \lambda\, u^p + u^q & x \in D \\ u = 0 & x \in \partial D, \end{cases}$$

where $D \subset \mathbb{R}^N$ is a bounded domain with smooth boundary ∂D, $0 < q < 1, 1 < p < \frac{N+\sigma}{N-\sigma}, N > \sigma, \lambda > 0$. To the best of our knowledge, fractional sublinear equations in the all $\mathbb{R}^N$, such as (1.1), have not been studied so far.

The analysis of the elliptic equation (1.3) is strictly related (see [12]) to the asymptotic behavior of solutions of the Cauchy problem for the porous medium equation with variable density ρ:

$$\begin{cases} \rho\, \partial_t u = \Delta\, [u^m] & x \in \mathbb{R}^N, \quad t > 0 \\ u = u_0 & x \in \mathbb{R}^N, \quad t = 0 \end{cases} \tag{1.5}$$

with $m = 1/\alpha, \rho > 0$. The question if analogous results hold for problem

$$\begin{cases} \rho\, \partial_t u + (-\Delta)^{\frac{\sigma}{2}}\, [u^m] = 0 & x \in \mathbb{R}^N, \quad t > 0 \\ u = u_0 & x \in \mathbb{R}^N, \quad t = 0 \end{cases} \tag{1.6}$$

is not the content of the present work, and still remains to be answered.

On the other hand, let us mention that in the following, we shall use existence and uniqueness results proved in [16] for problem (1.6) (see also [15]), in order to show uniqueness of solutions to (1.1).

We describe next how the paper is organized and outline the main contributions. In Section 2 we recall the needed mathematical background about the fractional Laplacian, its realization through the harmonic extension, both in bounded domains and in the whole space, and give the precise notion of solution we will deal with. As well as in [1], [2] we will consider energy solutions. In Section 3 we perform a detailed and self-contained analysis of the linear problem

$$(-\Delta)^{\frac{\sigma}{2}} U = \rho \qquad \text{in } \mathbb{R}^N, \tag{1.7}$$

establishing existence and uniqueness of solutions. Boundedness of solutions and behavior at infinity of solutions is investigated as well; in particular a decay estimate at infinity is shown using some results in [18]. In Section 4 we study existence of solutions to equation (1.1). More precisely, we prove that existence of bounded solutions to the linear equation (1.7) is sufficient (see Theorem 4.1) to existence of solutions to (1.1). This somehow rephrases, in the nonlocal framework, some results obtained in [3] for the local problem (1.3). Finally in Section 5, by exploiting uniqueness results for problem (1.6) proved in [16], we show uniqueness of solutions of (1.3) satisfying a decay condition at infinity; see Theorem 5.8.

2 Mathematical background

We always make the following assumption:

$$\begin{cases} \text{(i)} & \rho \in L^\infty(\mathbb{R}^N),\ \rho \geq 0 \text{ a.e. in } \mathbb{R}^N,\ \rho \not\equiv 0 \\ \text{(ii)} & 0 < \sigma < 2 \\ \text{(iii)} & 0 < \alpha < 1. \end{cases} \tag{$\mathbf{A}_0$}$$

Furthermore, about $\rho = \rho(x)$, we suppose that the following decay condition at infinity holds:

$$\begin{gathered} \text{there exist } \check{C} > 0, \check{R} > 0 \text{ and } \beta \in (N, \infty) \text{ such that} \\ \rho(x) \leq \check{C}|x|^{-\beta} \quad \text{for all } x \in \mathbb{R}^N \setminus B_{\check{R}}. \end{gathered} \tag{$\mathbf{A}_1$}$$

Let us introduce the following sets:

$$L^1_\rho(\mathbb{R}^N) := \left\{ f \text{ measurable in } \mathbb{R}^N \,\middle|\, \|f\|_{L^1_\rho(\mathbb{R}^N)} := \int_{\mathbb{R}^N} f\, \rho \,\mathrm{d}x < \infty \right\},$$

$$L^+_\rho(\mathbb{R}^N) := \left\{ f \in L^1_\rho(\mathbb{R}^N) \,\middle|\, f \geq 0 \right\}.$$

The fractional Laplace operator $(-\Delta)^{\sigma/2}$ can be defined in many different ways, one of which relies on the Fourier transform. For any g in the class of Schwartz functions, if $(-\Delta)^{\sigma/2} g = h$ then

$$\hat{h}(\xi) = |\xi|^\sigma \hat{g}(\xi). \tag{2.1}$$

When, as in the present situation, σ varies in the open interval $(0,2)$ then representation (1.2) holds.

Furthermore, if φ is a smooth and bounded function defined in $\mathbb{R}^N$, we can consider its σ–harmonic extension $v = \mathrm{E}(\varphi)$ to the upper half-space

$$\Omega := \mathbb{R}^{N+1}_+ = \{(x,y) :\, x \in \mathbb{R}^N,\, y > 0\},$$

that is, the unique smooth and bounded solution $v(x,y)$ of the problem

$$\begin{cases} \operatorname{div}\{y^{1-\sigma}\nabla v\} = 0 & \text{in } \Omega \\ v(x,0) = \varphi(x) & \text{in } \Gamma. \end{cases}$$

Here $\Gamma := \overline{\Omega} \cap \{y = 0\} \equiv \mathbb{R}^N$. It has been proved (see [7], [9]) that

$$-\frac{\partial v}{\partial y^\sigma}(x,0) = (-\Delta)^{\frac{\sigma}{2}}\,\varphi(x) \quad \text{for all } x \in \Gamma,$$

where

$$\mu_\sigma := \frac{2^{\sigma-1}\Gamma(\sigma/2)}{\Gamma(1-\sigma/2)},\ \frac{\partial v}{\partial y^\sigma}(x,0) := \mu_\sigma \lim_{y\to 0^+} y^{1-\sigma}\frac{\partial v}{\partial y}.$$

2.1 Problem in the all space

Multiplying the nonlocal partial differential equation in (1.1) by a test function φ compactly supported in $\mathbb{R}^N$, integrating by parts, taking into account (2.1) and using the Plancherel's Theorem, we discover that

$$\int_{\mathbb{R}^N} \rho\, u^\alpha \varphi\, \mathrm{d}x - \int_{\mathbb{R}^N} (-\Delta)^{\sigma/4} u\, (-\Delta)^{\sigma/4}\varphi\, \mathrm{d}x = 0. \tag{2.2}$$

We denote by $\dot{H}^{\sigma/2}(\mathbb{R}^N)$ the fractional Sobolev space obtained by completing $C_0^\infty(\mathbb{R}^N)$ with the norm $\|\psi\|_{\dot{H}^{\sigma/2}} = \|(-\Delta)^{\sigma/4}\psi\|_{L^2(\mathbb{R}^N)}$.

Definition 2.1. *A* solution *to equation* (1.1) *is a function* $u \geq 0$ *such that:*

- $u \in \dot{H}^{\sigma/2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $u^{1+\alpha} \in L^1_\rho(\mathbb{R}^N)$;
- *for any* $\varphi \in C_0^\infty(\mathbb{R}^N)$ *identity* (2.2) *holds.*

Solving problem (1.1) is equivalent to solve the following problem

$$\begin{cases} \operatorname{div}\{y^{1-\sigma}\nabla w\} = 0 & (x,y) \in \Omega, \\ \dfrac{\partial w}{\partial y^\sigma} = \rho u^\alpha & x \in \Gamma; \end{cases} \tag{2.3}$$

here $w = \mathrm{E}(u)$ is the σ-harmonic extension of u to the upper half-space Ω. We denote by $X^\sigma(\Omega)$ the completion of $C_0^\infty(\bar{\Omega})$ with the norm

$$\|v\|_{X^\sigma(\Omega)} = \left(\mu_\sigma \int_\Omega y^{1-\sigma}|\nabla v|^2\, \mathrm{d}x\, \mathrm{d}y\right)^{\frac{1}{2}}.$$

Given a function $f \in X^\sigma(\Omega)$ we denote by $f|_\Gamma$ its trace on Γ. We give next the definition of solution to problem (2.3).

Definition 2.2. *A* solution *to problem* (2.3) *is a pair of functions* (u, w), *with* $u \geq 0, w \geq 0$, *such that*

- $w \in X^\sigma(\Omega) \cap L^\infty(\Omega)$;
- $w|_\Gamma = u$, $u^{\alpha+1} \in L^1_\rho(\Gamma)$;
- *for any* $\varphi \in C_0^\infty(\bar{\Omega})$ *there holds*

$$\int_\Gamma \rho\, u^\alpha\, \varphi(x, 0)\, \mathrm{d}x = \mu_\sigma \int_\Omega y^{1-\sigma} \langle \nabla\varphi, \nabla w\rangle\, \mathrm{d}x\, \mathrm{d}y. \tag{2.4}$$

It is direct to see that the notion of solution given in Definition 2.1 for equation (1.1) is equivalent to that in Definition 2.2 for the extension problem (2.3) (see *e.g.* [9, Section 3.3] for a similar result). The same equivalence also holds for all other problems and equations we will consider in the sequel.

2.2 Problem in bounded domains

Let D be a bounded domain in $\mathbb{R}^N$ with smooth boundary ∂D. We use next a spectral decomposition to define the fractional operator $(-\Delta)^{\sigma/2}$ in D. Let $\{\xi_n\}_1^\infty$ be an orthonormal basis of $L^2(D)$ made by eigenfunctions of $-\Delta$ in D completed with homogeneous Dirichlet boundary conditions, and let $\{\lambda_n\}_1^\infty$ be the sequence of the corresponding eigenvalues. For any $u \in C_0^\infty(D)$

$$(-\Delta)^{\sigma/2} u := \sum_{n=1}^\infty \lambda_n^{\sigma/2}\, u_n\, \xi_n \quad \text{in } D\,,$$

where $u = \sum_{n=1}^\infty u_n\, \xi_n$ in $L^2(D)$. By density, $(-\Delta)^{\sigma/2} u$ can be also defined for u belonging to the Hilbert space

$$H_0^{\sigma/2}(D) := \left\{ u \in L^2(D) \;\middle|\; \|u\|^2_{H_0^{\sigma/2}(D)} := \sum_{n=1}^\infty \lambda_n^{\sigma/2} u_n^2 < \infty \right\}.$$

We then consider the problem

$$\begin{cases} (-\Delta)^{\frac{\sigma}{2}} u = \rho\, u^\alpha & \text{in } D \\ u = 0 & \text{in } \partial D. \end{cases} \tag{2.5}$$

Definition 2.3. *A* solution *to problem* (2.5) *is a function* $u \geq 0$ *such that:*

- $u \in H_0^{\sigma/2}(D)$, $u^{1+\alpha} \in L^1_\rho(D)$;
- *for any* $\varphi \in C_0^\infty(D)$ *identity* (2.2) *holds.*

As well as for problem (1.1), to solve problem (2.5) we can also consider the analogous of problem (2.3) in the half-cylinder $\mathcal{C}_D := D \times (0, \infty)$ with homogeneous zero conditions on the lateral boundary $\partial_L \mathcal{C}_D := \big\{(x, y) \in \Omega : x \in \partial D, y > 0\big\}$:

$$\begin{cases} \operatorname{div}\big\{y^{1-\sigma}\nabla w\big\} = 0 & (x,y) \in \mathcal{C}_D, \\ w = 0 & (x,y) \in \partial_L \mathcal{C}_D, \\ \dfrac{\partial w}{\partial y^\sigma} = \rho u^\alpha & x \in D. \end{cases} \tag{2.6}$$

We denote by $X_0^\sigma(\mathcal{C}_D)$ the closure of $C_0^\infty\big(D \times [0, \infty)\big)$ with respect to the norm

$$\|v\|_{X^\sigma(\mathcal{C}_D)} = \left(\mu_\sigma \int_{\mathcal{C}_D} y^{1-\sigma} |\nabla v|^2 \,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}}.$$

Definition 2.4. *A* solution *to problem* (2.6) *is a pair of functions* (u, w), *with* $u \geq 0, w \geq 0$, *such that*

- $w \in X_0^\sigma(\mathcal{C}_D)$*;*
- $w|_D = u,\ u^{\alpha+1} \in L^1_\rho(D)$*;*
- *for any* $\varphi \in C_0^\infty(\overline{\mathcal{C}_D}), \varphi = 0$ *on* $\partial_L \mathcal{C}_D$ *there holds*

$$\int_D \rho\, u^\alpha\, \varphi(x, 0)\,\mathrm{d}x = \mu_\sigma \int_{\mathcal{C}_D} y^{1-\sigma} \langle \nabla\varphi, \nabla w\rangle \,\mathrm{d}x\,\mathrm{d}y. \tag{2.7}$$

Remark 2.5. By the trace embedding theorem, if $w \in X_0^\sigma(\mathcal{C}_D)$, then $w \in L^p(D)$ whenever $1 \leq p \leq \frac{2N}{N-\sigma}$ and $N > \sigma$. Since $\sigma > 0$, then $p = 1 + \alpha$ belongs to $[1, \frac{2N}{N-\sigma}]$. Thus

$$\int_D \rho\, w^{1+\alpha}\,\mathrm{d}x \leq \|\rho\|_{L^\infty(D)} (\|w\|_{L^{1+\alpha}(D)})^{1+\alpha}.$$

In particular, the left hand side in (2.7) is finite. □

3 The linear problem

In this Section we study the linear problem

$$(-\Delta)^{\frac{\sigma}{2}} U = \rho \qquad \text{in } \mathbb{R}^N. \tag{3.1}$$

Definition 3.1. *A* solution *to problem* (3.1) *is a function* U *such that:*

- $U \in \dot{H}^{\sigma/2}(\mathbb{R}^N) \cap L^1_\rho(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$*;*

- *for any* $\varphi \in C_0^\infty(\mathbb{R}^N)$ *there holds*

$$\int_{\mathbb{R}^N} \rho\,\varphi\,\mathrm{d}x - \int_{\mathbb{R}^N} (-\Delta)^{\sigma/4} U\,(-\Delta)^{\sigma/4}\psi\,\mathrm{d}x = 0. \tag{3.2}$$

As well as for the nonlinear problem introduced in the previous Section, we consider problem

$$\begin{cases} \operatorname{div}\{y^{1-\sigma}\nabla W\} = 0 & (x,y) \in \Omega, \\ \dfrac{\partial W}{\partial y^\sigma} = \rho & x \in \Gamma, \end{cases} \tag{3.3}$$

for the harmonic extension $W = \mathrm{E}(U)$, and give the next

Definition 3.2. *A* solution *to problem* (3.3) *is a pair of functions* (U, W) *such that*

- $W \in X^\sigma(\Omega) \cap L^\infty(\Omega)$*;*
- $W|_\Gamma = U$, $U \in L^1_\rho(\mathbb{R}^N)$*;*
- *for any* $\varphi \in C_0^\infty(\bar\Omega)$ *there holds*

$$\int_\Gamma \rho\,\varphi(x,0)\,\mathrm{d}x = \mu_\sigma \int_\Omega y^{1-\sigma} \langle \nabla\varphi, \nabla W\rangle\,\mathrm{d}x\,\mathrm{d}y. \tag{3.4}$$

We also introduce the linear problem in a bounded domain $D \subset \mathbb{R}^N$,

$$\begin{cases} (-\Delta)^{\frac{\sigma}{2}} U = \rho & \text{in } D \\ U = 0 & \text{in } \partial D. \end{cases} \tag{3.5}$$

Definition 3.3. *A* solution *to problem* (3.5) *is a function* U *such that:*

- $u \in H_0^{\sigma/2}(D)$, $U \in L^1_\rho(D)$*;*
- *for any* $\varphi \in C_0^\infty(D)$ *identity* (3.2) *holds.*

The associated extension problem in the half-cylinder $\mathcal{C}_D := D \times (0, \infty)$ with zero lateral condition is:

$$\begin{cases} \operatorname{div}\{y^{1-\sigma}\nabla W\} = 0 & (x,y) \in \mathcal{C}_D, \\ W = 0 & (x,y) \in \partial_L \mathcal{C}_D, \\ \dfrac{\partial W}{\partial y^\sigma} = \rho & x \in D. \end{cases} \tag{3.6}$$

Definition 3.4. *A* solution *to problem* (3.6) *is a pair of functions* (U, W) *such that*

- $W \in X_0^\sigma(\mathcal{C}_D)$*;*

- *$W|_D = U$;*
- *for any $\varphi \in C_0^\infty(\overline{\mathcal{C}_D}), \varphi = 0$ on $\partial_L \mathcal{C}_D$, there holds*

$$\int_D \rho\,\varphi(x,0)\,\mathrm{d}x = \mu_\sigma \int_{\mathcal{C}_D} y^{1-\sigma} \langle \nabla\varphi, \nabla W\rangle\,\mathrm{d}x\,\mathrm{d}y. \tag{3.7}$$

We introduce the following property:

$$\text{there exits a solution } (U, W) \text{ to (3.3).} \tag{$\mathbf{H}$}$$

Remark 3.5. As described in Section 2.1, condition (**H**) is equivalent to the existence of a solution to problem (3.1), in the sense of Definition 3.1. □

3.1 Existence of solutions to the linear problem

Let

$$K^\sigma(x) := C_{N,\sigma} \frac{1}{|x|^{N-\sigma}} \quad (x \in \mathbb{R}^N \setminus \{0\})$$

be the Riesz kernel, where

$$C_{N,\sigma} := 2^{\sigma-1}\sigma \frac{\Gamma\big((N+\sigma)/2\big)}{\pi^{N/2}\Gamma\big((1-\sigma)/2\big)}\,. \tag{3.8}$$

As well as for the standard Laplace operator, a solution to problem (3.1) can be constructed by convolving such a kernel with the function ρ, that is

$$(K^\sigma * \rho)(x) = C_{N,\sigma} \int_{\mathbb{R}^N} \frac{\rho(y)}{|x-y|^{N-\sigma}}\mathrm{d}y\,.$$

We are interested in determining conditions to be imposed on ρ such that property (**H**) is satisfied.

Remark 3.6. (i) By results in [19], $\big(K^\sigma * \rho\big)(x)$ is finite at any $x \in \mathbb{R}^N$ if and only if

$$\int_{\mathbb{R}^N} \frac{|\rho(y)|}{1+|y|^{N-\sigma}}\mathrm{d}y < \infty \quad \text{and} \quad \int_{B(x,1)} \frac{|\rho(y)|}{|x-y|^{N-\sigma}}\mathrm{d}y \text{ for any } x \in \mathbb{R}^N.$$

(ii) As a consequence of (i), when $\rho \in L^\infty_{loc}(\mathbb{R}^N)$, $\big(K^\sigma * \rho\big)(x)$ is finite at any $x \in \mathbb{R}^N$ if and only if it is finite at some $x_0 \in \mathbb{R}^N$.

(iii) From (i) it immediately follows that if $\rho \in L^\infty(\mathbb{R}^N)$ and $\big(K^\sigma * \rho\big)(0) < \infty$, then $K^\sigma * \rho \in L^\infty(\mathbb{R}^N)$. In fact, in this case

$$\int_{B(x,1)} \frac{|\rho(y)|}{|x-y|^{N-\sigma}}\mathrm{d}y \leq \|\rho\|_{L^\infty(\mathbb{R}^N)} \int_{B(0,1)} \frac{\mathrm{d}z}{|z|^{N-\sigma}},$$

for any $x \in \mathbb{R}^N$.

□

From [18] the following lemma can be deduced (see [16, Corollary 5.4]).

Lemma 3.7. *Let $N \geq 2$. Let assumptions $(\mathbf{A}_0)$, $(\mathbf{A}_1)$ be satisfied. Then*

$$(K^\sigma * \rho)(x) \to 0 \quad as \ |x| \to \infty . \tag{3.9}$$

More precisely, for some $C > 0$, we have:

$$(K^\sigma * \rho)(x) \leq C|x|^{\sigma-\nu-\frac{N}{r}} \quad for\ all\ \ x \in \mathbb{R}^N, \tag{3.10}$$

provided $\frac{N}{2}(2-\sigma) < \nu < N$ and

$$\max\left\{\frac{2}{\sigma}, \frac{N}{\beta-\nu}\right\} < r < \frac{N}{N-\nu} . \tag{3.11}$$

Let assumption $(\mathbf{A}_0)$ be satisfied. Note that for each $R > 0$ problem

$$\begin{cases} (-\Delta)^{\frac{\sigma}{2}} U_R = \rho & \text{in } B_R \\ U_R = 0 & \text{in } \partial B_R. \end{cases} \tag{3.12}$$

admits a unique solution U_R; moreover, $U_R \geq 0$ in B_R, and

$$U_R(x) = \int_{B_R} G_R(x,y)\rho(y)\,\mathrm{d}y$$

where G_R is the Green function of the operator $(-\Delta)^{\sigma/2}$ in the domain B_R, completed with zero Dirichlet boundary conditions on ∂B_R.

Since $U_R \geq 0$ in B_R for any $R > 0$, if $R_1 < R_2$, then U_{R_2} is a supersolution to problem

$$\begin{cases} (-\Delta)^{\frac{\sigma}{2}} U = \rho & \text{in } B_{R_1} \\ U = 0 & \text{in } \partial B_{R_1}. \end{cases}$$

So, by comparison principles,

$$0 \leq U_{R_1} \leq U_{R_2} \quad \text{in } \ B_{R_1} .$$

By the monotone convergence of G_R to K^σ as $R \to \infty$, we have:

$$U_\infty(x) := \lim_{R\to\infty} U_R(x) = (K^\sigma * \rho)(x) \quad (x \in \mathbb{R}^N) . \tag{3.13}$$

We will denote by W_R the harmonic extension in $\mathcal{C}_R$ of U_R, and by W_∞ the harmonic extension in Ω of U_∞. Clearly, $W_\infty = \lim_{R\to\infty} W_R$. Some properties of the function $K^\sigma * \rho$ have been discussed in Remark 3.6. In next Proposition we give some criteria, involving the function $K^\sigma * \rho$, for existence of solutions to equation (3.1).

Proposition 3.8. *Assume $(\mathbf{A}_0)$–(i),(ii). Then property $(\mathbf{H})$ is satisfied if and only if*

$$K^\sigma * \rho \in L^\infty(\mathbb{R}^N) \cap L^1_\rho(\mathbb{R}^N).$$

Proof. Assume that property (**H**) holds and, without loosing generality, that the bounded solution (U,W) of (3.3) verifies $W\geq 0$ in Ω. Let (U_R,W_R) be solving problem (3.6) with $D=B_R$. By maximum principle,

$$0\leq U_R\leq U \quad \text{in } B_R. \tag{3.14}$$

By passing to the limit as $R\to\infty$, in view of (3.13), we get

$$U_\infty = K^\sigma * \rho \leq U \quad \text{in } \mathbb{R}^N\,. \tag{3.15}$$

By (**H**) and Definition 3.2, $U\in L^\infty(\mathbb{R}^N)\cap L^1_\rho(\mathbb{R}^N)$. Thus, from (3.14) and (3.15) we can infer that $K^\sigma*\rho\in L^\infty(\mathbb{R}^N)\cap L^1_\rho(\mathbb{R}^N)$.

On the other hand, suppose that $K^\sigma*\rho\in L^\infty(\mathbb{R}^N)\cap L^1_\rho(\mathbb{R}^N)$. By (3.13), there exists $\tilde{C}>0$ such that for any $R>0$

$$0\leq U_R\leq \tilde{C} \quad \text{in } B_R. \tag{3.16}$$

For each $R>0$, choose a sequence $\{\varphi_n\}\subset C_0^\infty\big(B_R\times[0,\infty)\big), \varphi_n\to W_R$ in $X_0^\sigma(\mathcal{C}_R)$ as $n\to\infty$. So, for each $R>0$, for all $n\in\mathbb{N}$ there holds:

$$\mu_\sigma\int_{\mathcal{C}_R} y^{1-\sigma}\langle\nabla W_R,\nabla\varphi_n\rangle\,\mathrm{d}x\,\mathrm{d}y = \int_{B_R}\rho\,\varphi_n(x,0)\,\mathrm{d}x\,;$$

letting $n\to\infty$ we have:

$$\mu_\sigma\int_{\mathcal{C}_R} y^{1-\sigma}|\nabla W_R|^2\,\mathrm{d}x\,\mathrm{d}y = \int_{B_R}\rho\,U_R\,\mathrm{d}x\,. \tag{3.17}$$

Let $V\subset\subset\bar{\Omega}$. Take $R_0>0$ so that $V\subset\mathcal{C}_{R_0}$. Hence, for every $R>R_0$, Since $K^\sigma*\rho\in L^1_\rho(\mathbb{R}^N)$, using monotone convergence theorem and (3.13), we have:

$$\int_V \rho\,U_R\,\mathrm{d}x \leq \|K^\sigma*\rho\|_{L^1_\rho(\mathbb{R}^N)}. \tag{3.18}$$

From (3.17) and (3.18) we get, for every $R>R_0$,

$$\mu_\sigma\int_V y^{1-\sigma}|\nabla W_R|^2\,\mathrm{d}x\,\mathrm{d}y\leq C, \tag{3.19}$$

for some C independent of R and V. Therefore, passing to the limit as $R\to\infty$, we get that (U_∞,W_∞) is a solution to problem (3.3). This completes the proof. □

Corollary 3.9. *If property* (**H**) *holds, then* (U_∞,W_∞) *is the minimal positive solution of* (3.3).

Proof. The statement follows by observing that in the proof of Proposition 3.8, U can be any nonnegative solution to (3.1). □

Remark 3.10. From Remark 3.6, it immediately follows that if $(\mathbf{A}_0)$, $(\mathbf{A}_1)$ are satisfied, then $K^\sigma*\rho\in L^\infty(\mathbb{R}^N)\cap L^1_\rho(\mathbb{R}^N)$. □

3.2 Uniqueness for the linear problem

Lemma 3.11. *Let assumptions* $(\mathbf{A}_0)$, $(\mathbf{A}_1)$ *be satisfied. If* U *is a bounded solution to problem* (3.1), *such that* $U(x) \to 0$ *as* $|x| \to \infty$, *then it coincides with* U_∞.

Proof. Let W, W_∞ be the extension in Ω of U and U_∞, respectively. Set $\tilde{W} := W - W_\infty$. Take a sequence $\varphi_n \subset C_0^\infty(\bar{\Omega})$ such that $\varphi_n \to \tilde{W}$ as $n \to \infty$ in $X_0^\sigma(\Omega)$. By Definition 3.2, for all $n \in \mathbb{N}$, we have

$$\int_\Omega \langle \nabla \tilde{W}, \nabla \varphi_n \rangle \mathrm{d}x\, \mathrm{d}y = 0.$$

Sending $n \to \infty$, we get

$$\int_\Omega |\nabla \tilde{W}|^2 \mathrm{d}x\, \mathrm{d}y = 0,$$

so $\tilde{W}$ is constant in Ω. Furthermore, in view of (3.9) and taking into account that by assumption $U(x) \to 0$ as $|x| \to \infty$, we deduce that $\tilde{W}(x, 0) \to 0$ as $|x| \to \infty$. This implies the identity $\tilde{W} \equiv 0$ and thus the statement. □

Lemma 3.12. *Let assumptions* $(\mathbf{A}_0)$, $(\mathbf{A}_1)$ *be satisfied. Let* U *be a solution to*

$$(-\Delta)^{\sigma/2} U \le \rho \qquad \textit{in } \mathbb{R}^N \tag{3.20}$$

such that $U(x) \to 0$ *as* $|x| \to \infty$. *In addition, suppose that* $f := (-\Delta)^{\sigma/2} U \in L^\infty(\mathbb{R}^N)$. *Then*

$$U \le U_\infty \qquad \textit{in } \mathbb{R}^N. \tag{3.21}$$

Proof. Let $g := (-\Delta)^{\sigma/2}(U_\infty - U)$. Thus

$$g \ge 0 \qquad \text{in } \mathbb{R}^N. \tag{3.22}$$

Consider the equation

$$(-\Delta)^{\sigma/2} V = g \qquad \text{in } \mathbb{R}^N\,. \tag{3.23}$$

Note that $g = \rho - f$. Thus, in view of hypothesis $(\mathbf{A}_1)$ and the fact that $f \in L^\infty(\mathbb{R}^N)$, from Remark 3.6 we can infer that $K^\sigma * g \in L^\infty(\mathbb{R}^N)$. Furthermore, since $0 \le g \le \rho$, from Remark 3.10, it also follows that $K^\sigma * g \in L^1_g(\mathbb{R}^N)$.

So, by Proposition 3.8, $K^\sigma * g$ is a bounded solution to equation (3.23). Since $g \le \rho$, from Lemma 3.7 and (3.22) it follows that $\big(K^\sigma * g\big)(x) \to 0$ as $|x| \to \infty$. Clearly, $U_\infty - U$ is a bounded solution to equation (3.23) such that $U_\infty(x) - U(x) \to 0$ as $|x| \to \infty$. From Lemma 3.11 we deduce that

$$U_\infty - U = K^\sigma * g \quad \text{in } \mathbb{R}^N\,.$$

Now, from (3.22) the conclusion follows. □

4 Existence results

Our goal is to prove the following:

Theorem 4.1. *Let assumption* $(\mathbf{A}_0)$ *be satisfied. If* $K^\sigma * \rho \in L^\infty(\mathbb{R}^N) \cap L^1_\rho(\mathbb{R}^N)$*, then there exists a solution to problem* (2.3).

Remark 4.2. By results in Section 2.1, we can reformulate Theorem 4.1 as follows. If $K^\sigma * \rho \in L^\infty(\mathbb{R}^N) \cap L^1_\rho(\mathbb{R}^N)$, then there exists a solution to problem (1.1). □

In the sequel, for any $R > 0$, we shall make use of problem

$$\begin{cases} \operatorname{div}\{y^{1-\sigma}\nabla w_R\} = 0 & (x,y) \in \mathcal{C}_R := B_R \times (0,\infty), \\ w_R = 0 & x \in \partial\mathcal{C}_R,\, y > 0, \\ \dfrac{\partial w_R}{\partial y^\sigma} = \rho u_R^\alpha & x \in B_R. \end{cases} \tag{4.1}$$

Now, we state some results concerning problem (4.1), that can be proved by standard methods (see [2]).

Proposition 4.3. *Let assumption* $(\mathbf{A}_0)$ *be satisfied. Then for any* $R > 0$ *there exists a solution* (u_R, w_R) *to problem* (4.1).

Proof. Consider the functional $J : X_0^\sigma(\mathcal{C}_R) \to \mathbb{R}$ defined as

$$J(w) := \frac{1}{2}\int_{\mathcal{C}_R} y^{1-\sigma}\,|\nabla w|^2 \,\mathrm{d}x\,\mathrm{d}y - \frac{1}{\alpha+1}\int_{B_R} \rho\, w^{\alpha+1}\,\mathrm{d}x\,.$$

In view of Remark 2.5, it is well defined, bounded from below and coercive in $X_0^\sigma(\mathcal{C}_R)$. Then by standard tools, the conclusion follows. □

By the classical procedure of sub– and super solutions, next Lemma can be deduced (see [2, Lemma 4.2]).

Lemma 4.4. *Let assumption* $(\mathbf{A}_0)$ *be satisfied. Let* (u_1, w_1) *and* (u_2, w_2) *be respectively a subsolution and a supersolution to problem* (2.6)*, and assume that* $w_1 \le w_2$ *in* $\mathcal{C}_D$*. Then there exists* (u, w) *solution to problem* (2.6) *such that* $w_1 \le w \le w_2$ *in* $\mathcal{C}_D$.

The following comparison result can be easily deduced by the same arguments as in [2, Lemma 4.3] .

Lemma 4.5. *Let assumption* $(\mathbf{A}_0)$ *be satisfied. Let* (u_1, w_1) *and* (u_2, w_2) *be respectively a subsolution and a supersolution to problem* (2.6)*, and assume that* $w_1, w_2 > 0$*. Then* $w_1 \le w_2$ *in* $\mathcal{C}_D$.

Now we can prove Theorem 4.1.

Proof of Theorem 4.1. Since $K^\sigma * \rho \in L^\infty(\mathbb{R}^N) \cap L^1_\rho(\mathbb{R}^N)$, by Proposition 3.8 there exists (U, W) solution to problem (3.3). By strong maximum principle,

$$w_R > 0 \quad \text{in } \mathcal{C}_R, \;\; u_R > 0 \quad \text{in } \; B_R \,. \tag{4.2}$$

For any $R > 0$, by Proposition 4.3 a solution (u_R, w_R) to problem (4.1) exists. Moreover, by Lemma 4.5 it is unique. Observe that

$$R < R' \quad \Rightarrow \quad w_R \leq w_{R'} \text{ in } \mathcal{C}_R. \tag{4.3}$$

In fact, in view of (4.2), $(u_{R'}, w_{R'})$ is a supersolution to (4.1). Then $w_R \leq w_{R'}$. Moreover, for $C \geq \|U\|^{\frac{\alpha}{1-\alpha}}_{L^\infty(B_R)}$, (CU, CW) is a supersolution to problem (4.1). In fact, for any $\varphi \in C^\infty_0(\bar{\Omega})$,

$$\mu_\sigma \int_\Omega y^{1-\sigma} \langle \nabla\varphi, \nabla(CW)\rangle \,\mathrm{d}x\,\mathrm{d}y = \int_\Gamma C\,\rho\,\varphi(x,0)\,\mathrm{d}x \geq \int_\Gamma \rho\,(CU)^\alpha\,\varphi(x,0)\,\mathrm{d}x.$$

Hence,

$$0 \leq w_R \leq CW \quad \text{in } \mathcal{C}_R. \tag{4.4}$$

From (4.3) and (4.4) it follows that there exist

$$w := \lim_{R\to\infty} w_R \quad \text{in } \Omega,$$
$$u := \lim_{R\to\infty} u_R \quad \text{in } \Gamma.$$

Furthermore, $w \in L^\infty(\Omega), u \in L^\infty(\Gamma)$.

For each $R > 0$, take a sequence $\{\varphi_n\} \subset C^\infty_0(B_R \times [0,\infty)), \varphi_n \to W_R$ in $X^\sigma_0(\mathcal{C}_R)$ as $n \to \infty$. From (2.7), for each $R > 0$, for all $n \in \mathbb{N}$, we have:

$$\mu_\sigma \int_{\mathcal{C}_R} y^{1-\sigma} \langle \nabla\varphi_n, \nabla W_R\rangle \,\mathrm{d}x\,\mathrm{d}y = \int_{B_R} \rho\, u_R^\alpha\, \varphi_n(x,0)\,\mathrm{d}x.$$

Letting $n \to \infty$ we obtain:

$$\mu_\sigma \int_{\mathcal{C}_R} y^{1-\sigma} |\nabla W_R|^2 \,\mathrm{d}x\,\mathrm{d}y = \int_{B_R} \rho\, u_R^{\alpha+1}\,\mathrm{d}x\,. \tag{4.5}$$

Take any open subset V with $\bar{V} \subset \bar{\Omega}$ and select $R_0 > 0$ so big that $V \subset \mathcal{C}_{R_0}$. From (4.4) and (**H**), since $U \in L^1_\rho(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, there exists $C > 0$, independent of R, such that

$$\int_{B_R} \rho\, u_R^{\alpha+1}\,\mathrm{d}x \leq C \qquad \text{for any } R > 0. \tag{4.6}$$

Therefore (2.7) implies

$$\mu_\sigma \int_{\mathcal{C}_R} y^{1-\sigma} |\nabla w_R|^2 \mathrm{d}x\,\mathrm{d}y \leq C \qquad \text{for any } R > 0. \tag{4.7}$$

By letting $R \to \infty$ in (2.7), from (4.6) and (4.7) we obtain (2.4), $w \in X^\sigma(\Omega) \cap L^\infty(\Omega)$ and $u^{\alpha+1} \in L^1_\rho(\Gamma)$. $\square$

Remark 4.6. In the proof of Theorem 4.1 we have constructed a solution (u, w). Such solution turns out to be *minimal*, in the sense that if $(\tilde{u}, \tilde{w})$ is another solution, then $u \leq \tilde{u}$ and $w \leq \tilde{w}$. Moreover, by construction it follows that $u > 0$ in Γ, $w > 0$ in Ω. □

Remark 4.7. Assume that ρ satisfies $(\mathbf{A}_1)$. Then the minimal solution constructed in Theorem 4.1 satisfies the following identity:

$$u(x) = \int_{\mathbb{R}^N} \frac{\rho(y)\, u^\alpha(y)}{|x-y|^{N-\sigma}} \mathrm{d}y.$$

In fact, let $f(x) := \rho(x) u^\alpha(x)$ $(x \in \mathbb{R}^N)$, and consider the equation

$$(-\Delta)^{\sigma/2} v = f \quad \text{in } \mathbb{R}^N. \tag{4.8}$$

Since $u \in L^\infty(\mathbb{R}^N)$ and ρ satisfies $(\mathbf{A}_1)$, by Remark 3.10, $v = K^\sigma * f \in L^\infty(\mathbb{R}^N) \cap L^1_\rho(\mathbb{R}^N)$. By Proposition 3.8, v is a solution to equation (4.8). Since u is nonnegative and bounded, from Lemma 3.7 it follows that $v(x) \to 0$ as $|x| \to \infty$. Clearly, the same holds for u, u being the minimal solution to the same equation. Thus, from Lemma 3.11 the conclusion follows. □

Remark 4.8. The dependence of the solution of problem (1.1) upon ρ is monotone increasing. In fact, if $\rho_1 \leq \rho_2$ and u_1 and u_2 are the corresponding solutions of (1.1), then u_2 is a supersolution to

$$\begin{cases} (-\Delta)^{\frac{\sigma}{2}} u = \rho_1\, u^\alpha & \text{in } B_R \\ u = 0 & \text{on } \partial B_R. \end{cases}$$

Thus the sequence $v_{1,R}$ of function approximating u_1 satisfy $v_{1,R} \leq u_2$ in B_R. Passing to the limit as $R \to \infty$ we get $u_1 \leq u_2$. □

5 Uniqueness results

5.1 Fractional porous medium equation with variable density

For later use we introduce next a fractional porous medium equation and recall some results established in [16]. Consider the following nonlinear non-local Cauchy problem:

$$\begin{cases} \rho\, \partial_t u + (-\Delta)^{\frac{\sigma}{2}} [u^m] = 0 & x \in \mathbb{R}^N, \quad t > 0 \\ u = u_0 & x \in \mathbb{R}^N, \quad t = 0. \end{cases} \tag{5.1}$$

The parameter m is greater or equal to 1, and we will take later $m = 1/\alpha$.

Definition 5.1. *A* solution *to problem* (5.1) *is a function* $u \geq 0$ *such that:*

- *$u \in C([0,\infty); L^1_\rho(\mathbb{R}^N))$ and $u^m \in L^2_{loc}((0,\infty) : \dot{H}^{\sigma/2}(\mathbb{R}^N))$;*
- *for any $T > 0$, $\psi \in C^1_0(\mathbb{R}^N \times (0,T))$ there holds*
$$\int_0^T \int_{\mathbb{R}^N} \rho\, u\, \partial_t \psi \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_{\mathbb{R}^N} (-\Delta)^{\sigma/4}(u^m)\,(-\Delta)^{\sigma/4}\psi \,\mathrm{d}x\,\mathrm{d}t = 0;$$
- *$u(\cdot,0) = u_0$ almost everywhere.*

Solving problem (5.1) is equivalent to solving the following quasi-stationary problem for $w = \mathrm{E}(u^m)$, with a dynamical boundary conditions:

$$\begin{cases} -\mathrm{div}\big\{y^{1-\sigma}\nabla w\big\} = 0 & (x,y) \in \Omega,\ t > 0 \\ \dfrac{\partial w}{\partial y^\sigma} = \rho\,\dfrac{\partial\left[w^{\frac{1}{m}}\right]}{\partial t} & x \in \Gamma,\ t > 0 \\ w = u_0^m & x \in \Gamma,\ t = 0. \end{cases} \tag{5.2}$$

Definition 5.2. *A* solution *to problem* (5.2) *is a pair of functions* (u,w) *with $u \geq 0$, $w \geq 0$, such that*

- *$u \in C\big([0,\infty); L^1_\rho(\Gamma)\big)$;*
- *$w \in L^2_{loc}\big((0,\infty); X^\sigma(\Omega)\big)$;*
- *$w|_{\Gamma\times(0,\infty)} = u^m$;*
- *for any $T > 0, \psi \in C^1_0(\bar\Omega \times (0,T))$ there holds*
$$\int_0^T \int_\Gamma \rho\, u\, \partial_t \psi \,\mathrm{d}x\,\mathrm{d}t = \mu_\sigma \int_0^T \int_\Omega \nabla\psi \cdot \left(y^{1-\sigma}\nabla w\right)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t; \tag{5.3}$$
- *the identity $u(\cdot,0) = u_0$ holds almost everywhere.*

It is well known (see, *e.g.* [9, Section 3.3]) that a function u is a solution to problem (5.1) if and only if $(u, \mathrm{E}(u^m))$ is a solution to problem (5.2).

The previous definitions can be adapted to consider problem (5.1) in bounded domains. Let $R > 0$, $u_0 \in L^1_\rho(B_R)$ and consider the problem

$$\begin{cases} \rho\,\partial_t u + (-\Delta)^{\frac{\sigma}{2}}\,[u^m] = 0 & x \in B_R,\ t > 0, \\ u = 0 & x \in \partial B_R,\ t > 0, \\ u = u_0 & x \in B_R,\ t = 0. \end{cases} \tag{5.4}$$

Definition 5.3. *A* solution *to problem* (5.4) *is a function $u \geq 0$ such that:*

- *$u \in C([0,\infty); L^1_\rho(B_R))$ and $u^m \in L^2_{loc}((0,\infty) : \dot{H}^{\sigma/2}(B_R))$;*

- *for any* $T > 0, \psi \in C_0^1(B_R \times (0,T))$ *there holds*

$$\int_0^T \int_{B_R} \rho\, u\, \partial_t \psi \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{B_R} (-\Delta)^{\sigma/4} u^m \,(-\Delta)^{\sigma/4} \psi \,\mathrm{d}x\,\mathrm{d}t; \quad (5.5)$$

- $u(\cdot,0) = u_0$ *almost everywhere in* B_R.

Problem (5.4) can be solved through harmonic extensions considering, as discussed before for problem (5.1), the analogous of problem (5.2) in the half-cylinder $\mathcal{C}_R = B_R \times (0,\infty)$ with zero lateral condition:

$$\begin{cases} -\mathrm{div}\{y^{1-\sigma}\nabla w\} = 0 & (x,y) \in \mathcal{C}_R,\ t > 0; \\ w = 0 & x \in \partial\mathcal{C}_R,\ y > 0,\ t > 0; \\ \dfrac{\partial w}{\partial y^\sigma} = \rho\, \dfrac{\partial \left[|w|^{\frac{1}{m}}\right]}{\partial t} & x \in \mathcal{C}_R,\ y = 0,\ t > 0; \\ w = u_0^m & x \in \mathcal{C}_R, y = 0,\ t = 0. \end{cases} \quad (5.6)$$

Definition 5.4. *A* solution *to problem* (5.6) *is a pair of functions* (u, w), *with* $u \geq 0$, $w \geq 0$, *such that:*

- $u \in C\big([0,\infty); L^1_\rho(B_R)\big)$;
- $w \in L^2_{loc}\big((0,\infty); X_0^\sigma(\mathcal{C}_R)\big)$;
- $w|_{B_R \times (0,\infty)} = u^m$;
- *for any* $T > 0$ *and* $\psi = \psi(x,y,t)$, $\psi \in C_0^1(\bar{B}_R \times [0,\infty) \times (0,T)), \psi = 0$ *on* $\partial_L \mathcal{C}_R \times (0,T)$, *there holds*

$$\int_0^T \int_{B_R} \rho\, u\, \partial_t \psi \,\mathrm{d}x\,\mathrm{d}t = \mu_\sigma \int_0^T \int_{\mathcal{C}_R} \nabla\psi \cdot \big(y^{1-\sigma}\nabla w\big)\, \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t; \quad (5.7)$$

- *the identity* $u(\cdot,0) = u_0$ *holds almost everywhere in* B_R.

As well as in the case of $\mathbb{R}^N$ the two notions of solutions given in Definition 5.3 and Definition 5.4 are equivalent.

Observe that comparison principles hold for problem (5.4) (see [16]). Moreover, the existence of the minimal solution to problem (5.1) has been established in [16], together with some uniqueness results, among which we recall for later use the following:

Proposition 5.5. *Let* $N \geq 2$. *Let assumptions* $(\mathbf{A}_0)$, $(\mathbf{A}_1)$ *be satisfied. Moreover, suppose that* $\rho > 0$ *in* $\mathbb{R}^N$, $u_0 \in L^\infty(\mathbb{R}^N) \cap L^+_\rho(\mathbb{R}^N)$, $m \geq 1$. *Then there exists the minimal nonnegative solution* $\underline{u}$ *to problem* (5.1); *moreover,*

$$\int_0^t \big(\underline{u}(x,s)\big)^m \mathrm{d}s \leq C|x|^{\sigma-\nu-\frac{N}{r}} \quad (5.8)$$

for almost every $x \in \mathbb{R}^N \setminus B_{\bar{R}}$ $(\bar{R} > 0)$, $t > 0$,
for some $C > 0$, *with* ν, r *as in Lemma 3.7.*

Furthermore, if u is a solution to problem (5.1) *such that* (5.8) *is satisfied with $\underline{u}$ replaced by u, then $u \equiv \underline{u}$.*

Proof. See [16, Theorem 5.9] □

5.2 Uniqueness of solutions for the elliptic problem

Lemma 5.6. *Let u_1 and u_2 respectively a subsolution and a supersolution of* (1.1)*. Then there exists u solution to* (1.1) *such that $u_1 \leq u \leq u_2$ in $\mathbb{R}^N$.*

Proof. Thanks to Lemma 3.12, we can apply the standard technique of monotone iteration in the whole $\mathbb{R}^N$, and get the conclusion (note that the same argument has been applied in the proof of [3, Theorem 2]). □

Lemma 5.7. *Let ρ_1 and ρ_2 satisfying* $(\mathbf{A}_0)$*–(i),* $(\mathbf{A}_1)$*, and assume $\rho_1 \leq \rho_2$. Then, for any u_1 bounded solution to*

$$-(\Delta)^{\frac{\sigma}{2}} u_1 = \rho_1\, u_1^{\alpha} \qquad \textit{in } \mathbb{R}^N$$

there exists u_2 bounded solution to

$$(-\Delta)^{\frac{\sigma}{2}} u_2 = \rho_2\, u_2^{\alpha} \qquad \textit{in } \mathbb{R}^N \tag{5.9}$$

such that

$$u_2(x) \leq C|x|^{\sigma-\nu-\frac{N}{r}}, \qquad u_1 \leq u_2 \;\textit{ in } \mathbb{R}^N \tag{5.10}$$

for some $C > 0$, with ν and r as in Lemma 3.7.

Proof. Set

$$\tilde{C} = \left(\|u_1\|_{L^\infty(\mathbb{R}^N)}\right)^{\alpha}.$$

Then

$$(-\Delta)^{\frac{\sigma}{2}} u_1 \leq \rho_2\, u_1^{\alpha} \leq \tilde{C}\rho_2 \qquad \text{in } \mathbb{R}^N.$$

The function $V := \bar{C}(K^\sigma * \rho_2)$ satisfies, for $\bar{C} > \tilde{C}$ sufficiently large,

$$(-\Delta)^{\frac{\sigma}{2}} V = \bar{C}\rho_2 \geq \rho_2 V^{\alpha} \qquad \text{in } \mathbb{R}^N.$$

Thus u_1 and V are respectively a subsolution and a supersolution of the same problem:

$$(-\Delta)^{\frac{\sigma}{2}} U = \bar{C}\rho_2 \qquad \text{in } \mathbb{R}^N.$$

By Lemma 3.12, $u_1 \leq V$ in $\mathbb{R}^N$. Hence from Lemma 5.6 there exists a solution u_2 to problem (5.9) such that

$$u_1 \leq u_2 \leq V \quad \text{in } \mathbb{R}^N.$$

So, from Lemma 3.7 we get (5.10). This completes the proof. □

We establish first uniqueness under the stronger assumption that $\rho > 0$:

Proposition 5.8. *Assume* $(\mathbf{A}_0)$, $(\mathbf{A}_1)$. *Suppose further that* $\rho > 0$. *Let* $\underline{u}$ *be the minimal bounded solution to problem* (1.1) *provided by Theorem 4.1. Let* u *be any other bounded solution to problem* (1.1) *such that*

$$u^{\alpha}(x) \leq C|x|^{\sigma-\nu-\frac{N}{r}},$$

for some $C > 0$, *with* r *and* ν *as in Lemma 3.7. Then* $\underline{u} = u$ *in* $\mathbb{R}^N$.

Proof. Set $m := 1/\alpha$ and

$$C_m := (m-1)^{-\frac{1}{m-1}}.$$

Let $v_R(x,t)$ be the solution to

$$\begin{cases} \rho \, \dfrac{\partial v_R}{\partial t} + (-\Delta)^{\frac{\sigma}{2}} \left[v_R^m \right] = 0 & x \in B_R, \, t > 0, \\ v_R = 0 & x \in \partial B_R, \, t > 0, \\ v_R(x,0) = C_m u^{\frac{1}{m}} & x \in B_R. \end{cases} \tag{5.11}$$

Observe that the function

$$\tilde{u}(x,t) := \frac{C_m}{(t+1)^{\frac{1}{m-1}}} u^{\frac{1}{m}}(x)$$

solves

$$\rho \, \frac{\partial \tilde{u}}{\partial t} + (-\Delta)^{\frac{\sigma}{2}} \left[\tilde{u}^m \right] = 0, \qquad \text{in } \mathbb{R}^N \times (0,\infty).$$

Moreover, $\tilde{u}$ is a supersolution to problem (5.11). Thus, by comparison principles,

$$v_R \leq \tilde{u} \qquad \text{in } B_R \times (0,\infty). \tag{5.12}$$

Notice that for any $R > 0$

$$\operatorname*{ess\,inf}_{B_R} \underline{u} > 0.$$

Then we can select $\tau_R > 0$ such that

$$\frac{\underline{u}^{\frac{1}{m}}}{\tau_R^{\frac{1}{m-1}}} > u^{\frac{1}{m}} \qquad \text{in } B_R.$$

We have

$$\underline{\breve{u}}(x,t) := \frac{C_m \underline{u}^{\frac{1}{m}}}{(t+\tau_R)^{\frac{1}{m-1}}} \leq \frac{C_m \underline{u}^{\frac{1}{m}}}{t^{\frac{1}{m-1}}} =: \underline{\tilde{u}}(x,t) \qquad \text{in } B_R \times (0,\infty);$$

moreover $\underline{\breve{u}}$ is a supersolution to (5.11) thus, by comparison principles we get

$$v_R \leq \underline{\breve{u}} \leq \underline{\tilde{u}} \qquad \text{in } B_R \times (0,\infty). \tag{5.13}$$

Now, by results in [16] there exists the limit

$$v_\infty := \lim_{R\to\infty} v_R;$$

the function v_∞ solves

$$\begin{cases} \rho \dfrac{\partial v_\infty}{\partial t} + (-\Delta)^{\frac{\sigma}{2}}[v_\infty^m] = 0 & x \in \mathbb{R}^N,\ t>0,\\ v_R(x,0) = C_m u^{\frac{1}{m}} & x \in \mathbb{R}^N, \end{cases} \tag{5.14}$$

and satisfies the inequality

$$v_\infty^m(x,t) \le C|x|^{\sigma-\nu-\frac{N}{r}} \quad (x\in\mathbb{R}^N, t>0) \tag{5.15}$$

for some $C>0$, with ν and r as in Lemma 3.7. Then, by passing to the limit as $R\to\infty$ in (5.12),

$$v_\infty \le \tilde{u} \qquad \text{in } \mathbb{R}^N \times (0,\infty).$$

Notice that, as well as v_∞, the function $\tilde{u}$ solves (5.14) and satisfies the inequality (5.15). Then, by Proposition 5.5

$$v_\infty = \tilde{u} \qquad \text{in } \mathbb{R}^N \times (0,\infty).$$

Passing to the limit as $R\to\infty$ in (5.13), we obtain

$$v_\infty \le \underline{\tilde{u}} \qquad \text{in } \mathbb{R}^N \times (0,\infty)$$

which in turns entails

$$\frac{u^{\frac{1}{m}}}{\underline{u}^{\frac{1}{m}}} \le \frac{(t+1)^{\frac{1}{m-1}}}{t^{\frac{1}{m-1}}}.$$

As $t\to+\infty$ we get

$$u^{\frac{1}{m}} \le \underline{u}^{\frac{1}{m}} \qquad \text{in } \mathbb{R}^N.$$

Since $\underline{u}$ is minimal it follows that $u=\underline{u}$. □

We discuss now the general case in which $\rho \ge 0$.

Theorem 5.9. *Assume* $(\mathbf{A}_0)$, $(\mathbf{A}_1)$. *Let* $\underline{u}$ *and* u *be as in Proposition 5.8. Then* $\underline{u}=u$ *in* $\mathbb{R}^N$.

Proof. Let $h\in C^\infty(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)\cap L^1(\mathbb{R}^N)$, $h>0$, and define for any $\epsilon>0$,

$$\rho_\epsilon := \rho + \epsilon h.$$

By Lemma 5.7 there exists u_ϵ solving

$$-(\Delta)^{\frac{\sigma}{2}} u_\epsilon = \rho_\epsilon\, u_\epsilon^\alpha \qquad \text{in } \mathbb{R}^N \tag{5.16}$$

and verifying the following inequalities in $\mathbb{R}^N$:

$$\begin{aligned} u_\epsilon(x) &\le C|x|^{\sigma-\nu-\frac{N}{r}}, \\ u &\le u_\epsilon, \end{aligned} \tag{5.17}$$

for some $C>0$, with ν and r as in Lemma 3.7. Thanks to Proposition 5.8 such u_ϵ is the unique solution of (5.16). Let $(u_{\epsilon,R}, w_{\epsilon,R})$ and (u_R, w_R) be the positive solutions of the extension problems associated respectively to

$$\begin{cases} (-\Delta)^{\frac{\sigma}{2}} u_{\epsilon,R} = \rho_\epsilon\, u_{\epsilon,R}^\alpha & \text{in } B_R \\ u_{\epsilon,R} = 0 & \text{in } \partial B_R, \end{cases} \tag{5.18}$$

and

$$\begin{cases} (-\Delta)^{\frac{\sigma}{2}} u_R = \rho\, u_R^\alpha & \text{in } B_R \\ u_R = 0 & \text{in } \partial B_R. \end{cases} \tag{5.19}$$

By (2.7), for any $\varphi \in C_0^\infty(\overline{\mathcal{C}_R}), \varphi = 0$ on $\partial_L \mathcal{C}_R$,

$$\int_{B_R} \rho_\epsilon\, u_{\epsilon,R}^\alpha\, \varphi(x,0)\, \mathrm{d}x = \mu_\sigma \int_{\mathcal{C}_R} y^{1-\sigma} \langle \nabla\varphi, \nabla w_{\epsilon,R}\rangle\, \mathrm{d}x\, \mathrm{d}y; \tag{5.20}$$

$$\int_{B_R} \rho\, u_R^\alpha\, \varphi(x,0)\, \mathrm{d}x = \mu_\sigma \int_{\mathcal{C}_R} y^{1-\sigma} \langle \nabla\varphi, \nabla w_R\rangle\, \mathrm{d}x\, \mathrm{d}y. \tag{5.21}$$

It is easily seen that (5.20) holds true with $\varphi = w_R$, while (5.21) holds true with $\varphi = w_{\epsilon,R}$; so, we obtain:

$$\begin{aligned} \int_{B_R} \rho_\epsilon\, u_{\epsilon,R}^\alpha\, u_R(x)\, \mathrm{d}x &= \mu_\sigma \int_{\mathcal{C}_R} y^{1-\sigma} \langle \nabla w_{\epsilon,R}, \nabla w_R\rangle\, \mathrm{d}x\, \mathrm{d}y \\ &= \int_{B_R} \rho\, u_R^\alpha\, u_{\epsilon,R}(x)\, \mathrm{d}x. \end{aligned} \tag{5.22}$$

Hence,

$$\begin{aligned} \int_{B_R} \rho\, u_{\epsilon,R} u_R^\alpha (u_{\epsilon,R}^{1-\alpha} - u_R^{1-\alpha})\, \mathrm{d}x &= \int_{B_R} \rho \big[u_{\epsilon,R}\, u_R^\alpha - u_{\epsilon,R}^\alpha\, u_R\big] \mathrm{d}x \\ &= \int_{B_R} \big[\rho_\epsilon\, u_{\epsilon,R}^\alpha\, u_R - \rho\, u_{\epsilon,R}^\alpha\, u_R\big] \mathrm{d}x \\ &= \int_{B_R} (\rho_\epsilon - \rho)\, u_{\epsilon,R}^\alpha\, u_R\, \mathrm{d}x \\ &\le \int_{B_R} \epsilon\, h\, u_{\epsilon,R}^\alpha\, u_R\, \mathrm{d}x \le C\epsilon \|h\|_{L^1(\mathbb{R}^N)} \le C\epsilon \end{aligned}$$

for some $C>0$ independent of R. Passing to the limit as $R \to \infty$ and taking

into account (5.17) we get

$$\int_{\mathbb{R}^N} \rho\, u^\alpha\, \underline{u}^\alpha\, (u^{1-\alpha} - \underline{u}^{1-\alpha})\, \mathrm{d}x \le \int_{\mathbb{R}^N} \rho\, u_\epsilon^\alpha\, \underline{u}^\alpha\, (u_\epsilon^{1-\alpha} - \underline{u}^{1-\alpha})\, \mathrm{d}x$$
$$\le \lim_{R\to\infty} \int_{B_R} \rho\, u_{\epsilon,R}\, u_R^\alpha (u_{\epsilon,R}^{1-\alpha} - u_R^{1-\alpha})\, \mathrm{d}x \le C\epsilon. \quad (5.23)$$

Since $u \ge \underline{u}$, by sending $\epsilon \to 0^+$ in (5.23) we discover

$$\int_{\mathbb{R}^N} \rho\, u^\alpha\, \underline{u}^\alpha\, (u^{1-\alpha} - \underline{u}^{1-\alpha})\, \mathrm{d}x = 0.$$

Hence $\rho u^\alpha = \rho \underline{u}^\alpha$ in $\mathbb{R}^N$, which implies

$$(-\Delta)^{\frac{\sigma}{2}} (u - \underline{u}) = \rho\, (u^\alpha - \underline{u}^\alpha) = 0 \qquad \text{in } \mathbb{R}^N.$$

By uniqueness of solutions for the linear problem (see Lemma 3.11), we conclude that $u = \underline{u}$ in $\mathbb{R}^N$. $\square$

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